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The solitary wave of maximum amplitude

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The maximum amplitude of the solitary wave of constant form is determined to be $0{\cdot}83b$, where b is the depth far from the crest. In the analysis it is assumed that the crest is pointed and the motion is two-dimensional and irrotational. The complex velocity potential is expressed in terms of known singularities and an infinite power series with unknown coefficients. Approximate solutions are obtained by truncating the power series after N terms, where $N = 1, 3, 5, 7$, and 9. The amplitude, a measure of the error, and several other pertinent quantities are computed for each value of N.

1. Introduction

Since the experimental studies of Russell (1844) there has been intense interest in the motion of gravity waves. Of this interest no small part has been focused on the solitary wave. Russell described solitary waves which moved with almost constant form slowly decreasing in amplitude. For some time it was not known if this decrease in amplitude was due completely or only in part to viscous resistance. In particular, the question was, does there exist a two-dimensional, irrotational solitary wave which is of truly constant form and celerity? Stokes (1891) at one time argued that there did not exist such a solution but later reversed this opinion. McCowan (1891) and others, by obtaining various approximate solutions, substantiated the latter opinion of Stokes. Finally, in 1954, Friedrichs & Hyers gave an existence proof. This proof, however, did not give any insight as to what amplitudes these wave may take.

The experiments of Russell indicated that if a solitary wave exceeded a critical amplitude it would break. Rankine (1864) suggested that at this critical amplitude the speed of a particle at the crest was equal to the celerity of the wave, and the free surface formed a point or a cusp at the crest. Stokes (1880), by assuming this criterion, established the included angle of the point at the crest to be 120°. Approximate analyses by McCowan (1891) and Packham (1952) have since indicated that the critical amplitude does occur according to the Rankine criterion, but to the writer's knowledge, no proof has been presented.

A list of the symbols used throughout this paper is given below:

a	elevation of wave crest relative to channel bottom;
A	real-valued constant;
a_0, a_1, a_2, etc.	real-valued coefficients of a power series;
b	upstream depth;
b_1, b_2, b_3, etc.	real-valued coefficients of a power series;

c_0, c_1, c_2, etc.	real-valued coefficients;
$f(t)$	the transformation $z = f(t)$ which maps Γ onto the flow field;
Q	fluid discharge;
g	acceleration of gravity;
u	x-component of velocity;
U	upstream velocity;
v	y-component of velocity;
Γ	the domain $[\lvert t\rvert < 1, \operatorname{Im}(t) > 0]$;
F^2	defined by $F^2 = U^2/gb$;
ζ	the normalized complex velocity defined by $\zeta = (u-iv)/U$;
σ	real-valued variable;
η	real-valued variable;
χ	complex potential;
λ	defined by $\pi\lambda F^2 = \tan(\pi\lambda)$.

2. Problem definition

Because the wave moves with a uniform celerity, the fluid field may be made steady by viewing it from a co-ordinate system which moves with the wave. After this has been done, the flow field appears as shown in figure 1, where the

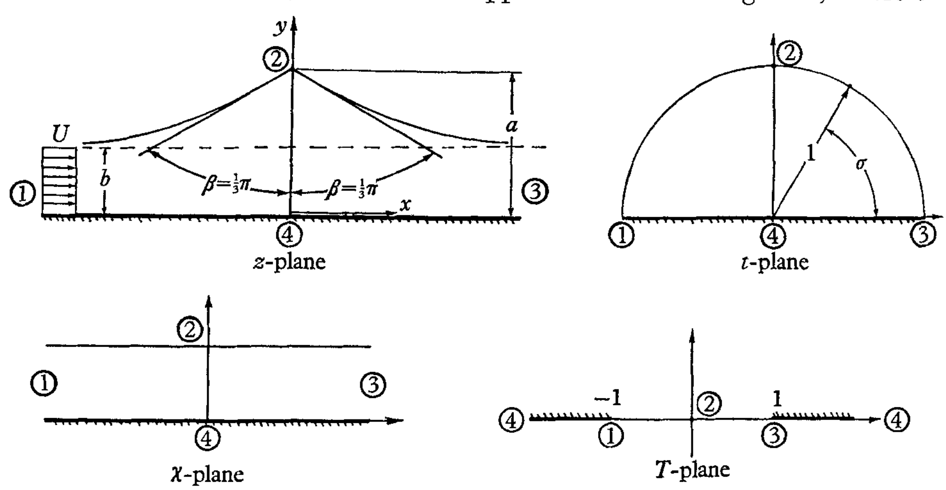

FIGURE 1. Physical and auxiliary planes.

crest is a stagnation point and the velocity U of the stream far from the crest is equal to the celerity of the wave. It will be assumed that the wave is symmetric about the y axis, that the fluid is inviscid, and that the motion is two-dimensional and irrotational.

A complex velocity potential $\chi(z)$ is sought which is analytic and possesses a non-vanishing derivative $d\chi/dz$ in the domain of the flow field (χ being the complex potential). Moreover, $d\chi/dz$ is to be continuous and non-vanishing on the boundary except for the stagnation point at (2). Lastly, χ is to be univalent in the domain of the flow field. If these conditions are satisfied, the complex velocity

$$\zeta(z) = (1/U)(u-iv),$$

which is related to χ by the expression

$$\zeta = \frac{1}{U}\frac{d\chi}{dz} \tag{1}$$

will be analytic in the domain of the flow field.

3. Solution

A necessary step in the solution is the mapping (in principle) of the domain $\Gamma = \{|t| < 1, \operatorname{Im}(t) > 0\}$ in the complex t-plane onto the domain of the flow field. The mapping $z = f(t)$ would be unique if three prescribed points on the boundary of the flow field were mapped into three prescribed points on the boundary $\overline{\Gamma}$ of the domain Γ. Hence, it is permissible to require the mapping of point (1) into $t = -1$, point (2) into $t = i$, and point (3) into $t = 1$. The function $f(t)$ exists and is continuous for $t\epsilon(\overline{\Gamma}, t \neq \pm 1)$ if χ exists. To see this we first note that $\chi(z)$ possesses an inverse $z(\chi)$ because it is univalent. This function $z(\chi)$ maps the strip $0 < \operatorname{Im}(\chi) < Q$ onto the domain of the flow field with the point $\chi = iQ$ going into $z = ai$. The sequence of mappings

$$\chi = -\frac{Q}{\pi}\log\left(\frac{T-1}{T+1}\right), \quad T = \frac{1}{2}\left(t+\frac{1}{t}\right) \tag{2), (3}$$

connects the strip $0 < \operatorname{Im}(\chi) < Q$ in the χ-plane to the domain Γ in the t-plane through the lower half of the intermediate T-plane. Because $f(t)$ is unique we must conclude that

$$f(t) = z\left\{\frac{2Q}{\pi}\log\left(\frac{1+t}{1-t}\right)\right\}, \tag{4}$$

which establishes the existence of $f(t)$. Moreover, since $z(\chi)$ is continuous on the boundary of the strip $0 < \operatorname{Im}(\chi) < Q$, $f(t)$ must be continuous for $t\epsilon(\overline{\Gamma}, t \neq \pm 1)$.

An equivalent statement of equation (4) is

$$\chi\{f(t)\} \equiv \chi(t) = \frac{2Q}{\pi}\log\left\{\frac{1+t}{1-t}\right\}, \tag{5}$$

which s the potential function for the domain Γ.

Behaviour of ζ near the crest

It is instructive to examine ζ in the neighbourhood of the point $T = 0$. Let

$$\zeta(T) = q^* e^{i\theta}, \tag{6}$$

where $q = Uq^*$ and θ is the argument of the velocity vector. From equation (2) we obtain

$$\frac{dz}{dT} = \frac{dz}{d\chi}\frac{d\chi}{dT} = \frac{dz}{d\chi}\frac{2Q}{\pi(1-T^2)}. \tag{7}$$

The constant-pressure condition on the free streamline, as expressed by the Bernoulli equation, requires

$$(q^2/2g) + y = a, \tag{8}$$

where q is the speed and y is the elevation of a point on the free streamline.

Differentiating equation (8) with respect to T and combining the resulting expression with equations (1), (6) and (7) produces

$$q^{*2}\frac{dq^*}{dT} = \frac{2}{\pi F^2}\frac{\sin\theta}{1-T^2}. \tag{9}$$

In addition to equation (9) q^* and θ must satisfy the conditions

$$\left.\begin{array}{ll} \left.\begin{array}{l} q^* \to 0 \\ \theta \to \frac{1}{2}\pi - \beta \end{array}\right\} & \text{as} \quad T \to +0, \\ \left.\begin{array}{l} q^* \to 0 \\ \theta \to -\frac{1}{2}\pi + \beta \end{array}\right\} & \text{as} \quad T \to -0. \end{array}\right\} \tag{10}$$

and

Equations (10) suggest that

$$\zeta \sim CT^{1-2\beta/\pi} e^{i(\frac{1}{2}\pi-\beta)} \tag{11}$$

as $T \to 0$ where C is a real constant. Substituting the above expression into the left- and right-hand sides of equation (9) gives

$$C^3 T^{2(1-2\beta/\pi)-2\beta/\pi}(1-2\beta/\pi)$$

and

$$\frac{2}{\pi F^2}\frac{\sin(\frac{1}{2}\pi-\beta)}{1-T^2} \quad \text{for} \quad T > 0.$$

Clearly equation (9) is satisfied by equation (11) as $T \to 0$ if

$$\beta = \tfrac{1}{3}\pi \tag{12}$$

and

$$C^3 = 3/(\pi F^2). \tag{13}$$

Hence, it is reasonable to assume that the solution $\zeta(T)$ can be expressed† as

$$\zeta = T^{\frac{1}{3}}F(T),$$

where

$$|F(0)| = (3/\pi F^2)^{\frac{1}{3}}.$$

By using equation (3) we obtain for the t-plane

$$\zeta(t) = (1+t^2)^{\frac{1}{3}}G(t), \tag{14}$$

where

$$|G(i)| = (3/2\pi F^2)^{\frac{1}{3}}. \tag{15}$$

The Ω function

From equation (14) we obtain

$$G(t) = \zeta(t)(1+t^2)^{-\frac{1}{3}}.$$

Because $\text{Im}[\zeta(t)] = 0$ $(-1 < t < 1)$ it can be seen from the above expression that $G(t)$ is real-valued and continuous here. Moreover, $G(t)$ is analytic for $t \epsilon \Gamma$ and non-vanishing for $t \epsilon \Gamma U \bar{\Gamma}$. Hence, by the reflexion principle $G(t)$ is analytic for $|t| < 1$ and non-vanishing for $|t| \leqslant 1$. Since it is convenient to work with the logarithm of $G(t)$ instead of $G(t)$ directly, we define

$$\Omega(t) = \log G(t) - \tfrac{1}{3}\log(2). \tag{16}$$

† A complete description of the singularity possessed by ζ at the crest is not attempted here. The interested reader is referred to Lewy (1950, 1952) and Carter (1961) for a discussion of closely related topics.

The function Ω will be analytic for $|t| < 1$ and continuous for $|t| \leqslant 1$ with the possible exception of $t = \pm 1$. Combining equations (14) and (16) produces

$$\zeta(t) = \{(1+t^2)^{\frac{1}{3}} e^{\Omega(t)}\}/2^{\frac{1}{3}}. \tag{17}$$

The free streamline

For $t = e^{i\sigma}$ $(0 \leqslant \sigma \leqslant \pi)$, let

$$\Omega(e^{i\sigma}) = \phi(\sigma) + i\epsilon(\sigma), \tag{18}$$

where ϕ and ϵ are real-valued functions of σ. Combining equations (17) and (18) gives

$$[q(\sigma)/U]^2 = \cos^{\frac{2}{3}}\sigma\, e^{2\phi} \quad (0 \leqslant \sigma < \tfrac{1}{2}\pi), \tag{19}$$

where $q(\sigma)$ is the speed at a point on the free streamline. Equation (5) gives

$$\chi(e^{i\sigma}) = (2Q/\pi)\,[\log(\cot\tfrac{1}{2}\sigma) + \tfrac{1}{2}\pi i],$$

or

$$\frac{d\chi}{d\sigma} = -\frac{2Q}{\pi}\frac{1}{\sin\sigma}. \tag{20}$$

Because

$$\frac{dz}{d\sigma} = \frac{dz}{d\chi}\frac{d\chi}{d\sigma} = \frac{d\chi}{d\sigma}\frac{1}{U\zeta}$$

we obtain by combining equations (14) and (17)

$$\frac{dz}{d\sigma} = -\frac{2Q}{\pi U}\frac{e^{-\phi - i(\sigma/3+\epsilon)}}{\sin\sigma\cos^{\frac{1}{3}}\sigma} \quad (0 \leqslant \sigma < \tfrac{1}{2}\pi)$$

or

$$y(\sigma) = a - \frac{2Q}{\pi U}\int_\sigma^{\frac{1}{2}\pi} \frac{e^{-\phi}\sin(\frac{1}{3}\xi + \epsilon)}{\sin\xi\cos^{\frac{1}{3}}\xi}\,d\xi, \tag{21}$$

and

$$x(\sigma) = \frac{2Q}{\pi U}\int_\sigma^{\frac{1}{2}\pi} \frac{e^{-\phi}\cos(\frac{1}{3}\xi + \epsilon)}{\sin\xi\cos^{\frac{1}{3}}\xi}\,d\xi. \tag{22}$$

Constant-pressure condition

Combining equations (19) and (21) with the Bernoulli equation [equation (8)] produces

$$\tfrac{1}{2}\cos^{\frac{2}{3}}\sigma\, e^{2\phi} = \frac{2}{\pi F^2}\int_\sigma^{\frac{1}{2}\pi} \frac{e^{-\phi}\sin(\frac{1}{3}\xi + \epsilon)}{\sin\xi\cos^{\frac{1}{3}}\xi}\,d\xi \quad (0 \leqslant \sigma < \tfrac{1}{2}\pi), \tag{23}$$

where $F^2 = U^3/Qg$ is the Froude number of the stream far from the crest. By differentiating and rearranging equation (23) we obtain

$$\phi' = \frac{2}{\pi F^2}\frac{e^{-3\phi}\sin(\frac{1}{3}\sigma + \epsilon)}{\sin\sigma\cos\sigma} + \frac{1}{3}\frac{\sin\sigma}{\cos\sigma} \quad (0 \leqslant \sigma < \tfrac{1}{2}\pi). \tag{24}$$

Combining equations (15), (17) and (18) gives

$$\frac{1}{\pi F^2}\, e^{-3\phi(\frac{1}{2}\pi)} = \tfrac{1}{3}. \tag{25}$$

Behaviour of Ω near $t = \pm 1$

The numerical solution obtained by De Boor (1961) for fluid flow under a sluice gate and those obtained by Watters & Street (1964) for flow over bumps in open channels indicate that $\zeta(t)$ is so badly behaved at points $t = \pm 1$ that it cannot be accurately approximated there by an analytic function. Apparently this difficulty occurs because $\zeta'(t)$ is not continuous at $t = \pm 1$. In this work part of this singularity is isolated and used to express $\zeta(t)$ in terms of a function $\Omega^*(t)$ which does have a continuous derivative at these points.

To study the form of this singularity it is convenient to work in the upper half of the W-plane with the points (1), (2) and (3) going into respectively the point at infinity, $W = 1$ and $W = 0$. Hence,

$$W = (1-T)/(T+1). \tag{26}$$

Let Δ be the domain $|W| < 1$, $\mathrm{Im}(W) > 0$. Because $\zeta(W)$ does not vanish for $W\epsilon\Delta$ we may define

$$\zeta(W) = e^{P(W)}, \tag{27}$$

where P is analytic for $W\epsilon\Delta$. Combining equations (9), (26) and (27) gives

$$\frac{dR}{dW} = -\frac{e^{-3R}\sin I}{W}\frac{1}{\pi F^2} \quad (0 \leqslant W < 1), \tag{28}$$

where R and I are real-valued for $W > 0$ and

$$P = R + iI. \tag{29}$$

Combining equations (7), (26), (27) and (29) produces

$$\frac{dZ}{dW} = -\frac{Q}{\pi U}\frac{e^{-R-iI}}{W}. \tag{30}$$

As a first attempt to determine the behaviour of $\zeta(W)$ for $W\epsilon\Delta$, let us assume

$$P(W) = A(e^{-\pi i}W)^{\lambda}, \tag{31}$$

where $\lambda > 0$ and A are real constants to be determined. Clearly $P(W)$ is real-valued for $T < 0$. For $T > 0$

$$R = AW^{\lambda}\cos\lambda\pi, \quad I = -AW^{\lambda}\sin\lambda\pi.$$

By substituting the above expressions into the left- and right-hand sides of equation (28) we obtain

$$dR/dW = \lambda AW^{\lambda-1}\cos\lambda\pi,$$

and

$$1/F^2\pi\,(e^{-3R}\sin I/W) = (1/F^2\pi)\,[AW^{\lambda-1}\sin\lambda\pi + O(W^{\lambda-1})],$$

where the symbol $O(W^{\lambda-1})$ indicates terms which approach zero as $W \to 0$ when divided by $W^{\lambda-1}$. Thus, it can be seen that equation (31) satisfies equation (28) as $W \to 0$ if

$$\tan\pi\lambda = \lambda\pi F^2. \tag{32}$$

Combining equations (30) and (31) and integrating the resulting expression gives

$$Z = -\frac{Q}{\pi U}\left[\log(W) + \frac{A(We^{-\pi i})^{\lambda}}{\lambda} + O(W^{\lambda})\right] + C.$$

For $W > 0$

$$X = -\frac{Q}{\pi U}\log(W) - \frac{AQ}{\lambda\pi U}W^{\lambda}\cos\lambda\pi + O(W^{\lambda}) + \mathrm{Re}(C),$$

and

$$y = \frac{AQ}{\lambda\pi U}W^{\lambda}\sin\lambda\pi + O(W^{\lambda}) + \mathrm{Im}(C)$$

for the equation of the free streamline. As $W \to 0$ $x \to \infty$ and $y \to \text{Im}(C)$. Clearly equation (31) represents the asymptotic behaviour of a jet on a flat plane, since y is monotonically increasing or decreasing depending upon the sign of A.

It was argued by Lenau (1965) that $P(W)$ possesses an asymptotic expansion of the form

$$P(W) \sim a_{1,0}(e^{-\pi i}W)^{\lambda} + a_{2,0}(e^{-\pi i}W)^{2\lambda} + \sum_{k=2}^{\infty}(e^{-\pi i}W)^{k\lambda}\sum_{m=0}^{k}a_{k,m}\log^m(e^{-\pi i}W)$$

at the point $W = 0$, where λ is a root of equation (32). These arguments which are long, tedious and not completely satisfactory will not be given here. From this expansion we note that

$$dP/dW \sim -\lambda a_{1,0}(e^{-\pi i}W)^{\lambda-1} - 2\lambda a_{2,0}(e^{-\pi i}W)^{2\lambda-1} + O(W^{2\lambda-1}).$$

Combining equations (3), (17), (26), and (27) with the above expression gives

$$\Omega'(t) + \frac{2}{3}\frac{t}{(1+t^2)} \sim -\frac{a_{1,0}(1-t)^{2\lambda-1}}{(4t)^{\lambda}t^{\lambda}}[2\lambda t^{\lambda} + (1-t)\lambda t^{\lambda-1}]$$
$$-\frac{a_{2,0}(1-t)^{4\lambda-1}}{(4t)^{2\lambda}t^{2\lambda}}[4\lambda t^{2\lambda} + (1-t)2\lambda t^{2\lambda-1}] + O[(1-t)^{4\lambda-1}]. \quad (33)$$

If $\lambda < \frac{1}{2}$ we observe from the above expansion that $\Omega'(t)$ is infinite at $t = 1$. Because the Froude number will be greater than unity, one sees from equation (32) that λ could lie in the interval $(0, \frac{1}{2})$. It has been assumed in this work that the only root λ of interest does lie in this interval.

The Ω^* *function*

Because of the method of obtaining approximate solutions used in this work, it is desirable to define a function Ω^* which has a continuous derivative at $t = \pm 1$. Hence, we define

$$\Omega^* = \Omega - \{(a_{1,0}(1-t^2)^{2\lambda})/(4^{\lambda})\,2\}. \quad (34)$$

Since the square of the Froude number should be about two, it will be assumed that $F^2 > 4/\pi$ so that $\frac{1}{4} < \lambda < \frac{1}{2}$. Thus, we have from equation (33)

$$\lim_{t\to 1}\Omega^*(t) = \lim_{t\to 1}\left[\Omega' + \frac{\lambda a_{1,0}(1-t^2)^{2\lambda-1}t^2}{4^{\lambda}}\right] = -\tfrac{1}{3}. \quad (35)$$

For $t = e^{i\sigma}$ $(0 \leqslant \sigma \leqslant \pi)$, let

$$\phi^*(\sigma) = \text{Re}(\Omega^*). \quad (36)$$

Then,

$$\epsilon^*(\sigma) = \text{Im}(\Omega^*),$$

and

$$\epsilon^{*\prime}(\sigma) = \text{Im}(d\Omega^*/d\sigma) = \text{Im}\{(d\Omega^*/dt)\,i\,e^{i\sigma}\}.$$

Hence, in view of equation (35),

$$\epsilon^{*\prime}(0) = -\tfrac{1}{3}. \quad (37)$$

From equation (34),

$$\Omega(t) = \Omega^*(t) + A(1-t^2)^{2\lambda}, \quad (38)$$

where A is a constant to be determined. Because Ω^* is real-valued for imaginary values of t, $\Omega^{*\prime}$ is not only continuous at $t = 1$ but by the reflexion principle must also be continuous at $t = -1$.

Because Ω and $(1-t^2)^{2\lambda}$ are analytic in $|t| < 1$ and real-valued for $-1 \leqslant t \leqslant 1$ it is seen from equation (38) that Ω^* must also have these properties. Thus, Ω^* may be expanded about the origin into a power series which has real coefficients, i.e.

$$\Omega^*(t) = a_0 + a_1 t + a_2 t^2 + \ldots,$$

where a_j $(j = 0, 1, 2, 3, 4, \ldots)$ are real numbers. However, Ω^* is also real-valued for $t = \xi i$, $0 \leqslant \xi < 1$. Thus, the coefficients of the odd powers of t must be zero so that

$$\Omega^*(t) = a_0 + a_2 t^2 + a_4 t^4 + \ldots |t| < 1. \tag{39}$$

For $t = e^{i\sigma}$ $(0 \leqslant \sigma \leqslant \frac{1}{2}\pi)$, we obtain from equations (29) and (27)

$$\left.\begin{aligned} \phi(\sigma) &= \phi^*(\sigma) + A\Gamma(\sigma), \\ \epsilon(\sigma) &= \epsilon^*(\sigma) + A\delta(\sigma), \end{aligned}\right\} \tag{40}$$

where $$\Gamma(\sigma) = \operatorname{Re}[(1-e^{2i\sigma})^{2\lambda}] = [2 - 2\cos 2\sigma]^{\lambda} \cos[\lambda(\pi - 2\sigma)] \tag{41}$$

and $$\delta(\sigma) = \operatorname{Im}[(1-e^{2i\sigma})^{2\lambda}] = -[2 - 2\cos 2\sigma]^{\lambda} \sin[\lambda(\pi - 2\sigma)]. \tag{42}$$

Combining equations (24), (25), (38), (39) and (40) produces

$$\begin{aligned} -\sum_{k=1}^{\infty} 2k(a_{2k} + Ab_{2k}) \sin(2k\sigma) = &-\frac{2}{\pi F^2} \exp\left\{-3\left[\sum_{k=0}^{\infty} a_{2k}\cos(2k\sigma) + A\Gamma(\sigma)\right]\right\} \\ &\times \frac{\sin\left[\sigma/3 + \sum_{k=1}^{\infty} a_{2k}\sin(2k\sigma) + A\delta(\sigma)\right]}{\sin\sigma\cos\sigma} \\ &+ \frac{1}{\pi F^2}\exp\left\{-3\left[\sum_{k=0}^{\infty} a_{2k}(-1)^k + A\Gamma(\tfrac{1}{2}\pi)\right]\right\}\frac{\sin\sigma}{\cos\sigma}, \end{aligned}$$

where $$b_{2k} = -\frac{2\lambda(1-2\lambda)(2-2\lambda)\ldots(k-1-2\lambda)}{k!} \quad (k = 1, 2, 3, \ldots).$$

Solving for the Fourier coefficients on the left side of the above expression produces the infinite system of equations

$$\begin{aligned} a_{2m} + Ab_{2m} = \frac{4}{\pi^2 F^2 m}\int_0^{\frac{1}{2}\pi} \Bigg\{ &\exp\left[-3\sum_{k=1}^{\infty} a_{2k}\cos(2k\sigma) - 3A\Gamma(\sigma)\right] \\ &\times \sin\left[\sigma/3 + \sum_{k=1}^{\infty} a_{2k}\sin(2k\sigma) + A\delta(\sigma)\right] \\ -\tfrac{1}{2}\exp\left[-3\sum_{k=0}^{\infty} a_{2k}(-1)^k - 3A\Gamma(\tfrac{1}{2}\pi)\right]\sin\sigma\Bigg\} &\frac{\sin(2m\sigma)}{\cos\sigma}\,d\sigma \quad (m = 1, 2, 3, \ldots). \end{aligned} \tag{43}$$

Combining equations (25), (39) and (40) produces

$$\frac{1}{\pi F^2}\exp\left\{-3\left[\sum_{k=0}^{\infty} a_{2k}(-1)^k + A\Gamma(\tfrac{1}{2}\pi)\right]\right\} = \frac{1}{3}. \tag{44}$$

By combining equations (37) and (39) we obtain

$$\sum_{k=1}^{\infty} 2k a_{2k} = -\tfrac{1}{3}. \tag{45}$$

Because $\zeta \to 1$ as $t \to \pm 1$, we obtain from equation (17)

$$\Omega(\pm 1) = 0,$$

and from equation (38)

$$\Omega^*(\pm 1) = 0.$$

Combining equation (39) with the above expression produces

$$a_0 = -\sum_{k=1}^{\infty} a_{2k}. \tag{46}$$

Numerical procedure used to obtain approximate solutions

Approximate solutions were obtained by truncating the power-series expansion of Ω^* to N terms and satisfying the first N of equations (43). By considering equations (25), (44), (45) and (46) in addition to these, one obtains $N+4$ non-linear simultaneous equations with $N+4$ unknowns a_j ($j = 0, 2, \ldots, 2N, A, \lambda, F^2$), This system of equations was reduced to an $N+3$ system by using equation (46) to eliminate the unknown a_0 from the remaining equations. Residuals $F_{k'}$ ($k = 1, 2, \ldots, N+3$) were defined such that

$$F_m = \frac{4}{\pi^2 F^2 m}\int_0^{\frac{1}{2}\pi}\left[\exp\left\{3\sum_{k=1}^{n}(1-\cos(2k\sigma))\,a_{2k} - 3A\Gamma(\sigma)\right\}\sin\left\{\tfrac{1}{3}\sigma + \sum_{k=1}^{n} a_{2k}\sin(2k\sigma)\right.\right.$$
$$\left.\left. + A\delta(\sigma)\right\} - \tfrac{1}{2}\exp\left\{3\sum_{k=1}^{n}(1-(-1)^k)\,a_{2k} - 3A\Gamma(\tfrac{1}{2}\pi)\right\}\sin\sigma\right]$$
$$\times\frac{\sin(2m\sigma)}{\cos\sigma}\,d\sigma - a_{2m} - Ab_{2m} \quad (m = 1, 2, 3, \ldots, N).$$

$$F_{n+1} = \frac{1}{3} + \sum_{k=1}^{n} 2ka_{2k},$$

$$F_{n+2} = \frac{1}{\pi F^2}\exp 3\left[\sum_{k=1}^{n}(1-(-1)^k)\,a_{2k} - 3A\Gamma(\tfrac{1}{2}\pi)\right] - \tfrac{1}{3},$$

$$F_{n+3} = \pi\lambda - \{\tan(\pi\lambda)/F^2\},$$

and the matrix equation

$$\begin{bmatrix} \dfrac{\partial F_1}{\partial a_2} & \dfrac{\partial F_1}{\partial a_4} & \cdots & \dfrac{\partial F_1}{\partial a_{2n}} & \dfrac{\partial F_1}{\partial A} & \dfrac{\partial F_1}{\partial(1/F^2)} & \dfrac{\partial F_1}{\partial \lambda} \\ \dfrac{\partial F_2}{\partial a_2} & & & & & & \\ \vdots & & & & & & \\ \dfrac{\partial F_{n+3}}{\partial a_2} & \cdots & & & & & \dfrac{\partial F_{n+3}}{\partial \lambda} \end{bmatrix}\begin{bmatrix} \Delta a_2 \\ \Delta a_4 \\ \vdots \\ \Delta(1/F^2) \\ \Delta\lambda \end{bmatrix} + \begin{bmatrix} F_1 \\ F_2 \\ \vdots \\ F_{n+3} \end{bmatrix} = 0,$$

was solved repeatedly for corrections $\Delta a_2, \Delta a_4, \ldots, \Delta\lambda$ until the residuals were small. For each cycle the most recent estimates of the unknowns were used to compute the residuals and their derivatives. The initial estimates were $a_{2k} = 0$ for $k = 2, 3, 4, \ldots, N$, $a_2 = -\frac{1}{6}$, $A = -0{\cdot}322$, $\lambda = 0{\cdot}322$ and $F^2 = 1{\cdot}60$. These estimates were obtained by assuming the amplitude of the wave to be $a/b = 1{\cdot}80$ and setting $a_{2k} = 0$ ($k = 2, 3, 4, \ldots, N$). From the Bernoulli equation one can obtain the relationship $a/b = 1 + \frac{1}{2}F^2$, which yields $F^2 = 1{\cdot}60$. Equation (45) gives $a_2 = -\frac{1}{6}$ and equation (32) gives $\lambda = 0{\cdot}322$. Finally, equation (44) yields, after some rearrangement,

$$A = -\frac{\log(\pi F^{\frac{2}{3}})^{-1}}{(3)\,2^{2\lambda}} = -0{\cdot}322.$$

Once the quantities a_{2j} ($j = 1, 2, \ldots, N, A, F^2$) and λ were determined, the approximation

$$\Omega(t) \cong -\sum_{k=1}^{n} a_{2k}(1-t^{2k}) + A(1-t^2)^{2\lambda}$$

was completely determined. The wave profile was then computed using equations (21) and (22). The amplitude, however, was determined from the expression

$$a = \frac{4Q}{\pi U}\int_0^1 \frac{e^{-\Omega(i\eta)}\,2^{\frac{1}{3}}}{(1+\eta^2)(1-\eta^2)^{\frac{1}{3}}}\,d\eta,$$

where $$\Omega(i\eta) = -\sum_{k=1}^{n} a_{2k}(1+\eta^{2k}) + A(1+\eta^2)^{2\lambda},$$

which can be obtained by employing equations (5), (17) and (34).

Error

Because the solutions obtained by the numerical procedure just discussed are not exact, there was a need for a measure of the error of a given solution. For this measure the equation

$$\tau(\sigma) = \tfrac{1}{2}\cos^{\frac{2}{3}}\sigma\, e^{2\phi} - \frac{2}{\pi F^2}\int_\sigma^{\frac{1}{2}\pi} \frac{e^{-\phi}\sin(\xi/3+\epsilon)}{\sin\xi\cos^{\frac{1}{3}}}\,d\xi$$

was used. From equation (20) we see that $\tau(\sigma) = 0$ for an exact solution, thus the magnitude of $|\tau|$ should be a measure of the accuracy of an approximate solution.

4. Discussion

From the computed results shown in table 1, it appears that good estimates of the wave amplitude a/b and of the upstream Froude number F^2 are $1{\cdot}83$ and $1{\cdot}65$, respectively. One can only guess to what accuracy these estimates are valid. Judging from the behaviour of $\|\tau\|$ and a/b with increasing N, it appears that the error is no more than $\pm 0{\cdot}01$.

In the section on the behaviour of ζ near $t = 1$, it was assumed that the proper value of λ lay in the interval $(0, \frac{1}{2})$. The behaviour of $\|\tau\|$ leaves little doubt as to the validity of this assumption.

Two previous attempts to determine the maximum amplitude of the solitary wave deserve special mention. First McCowan (1894) obtained the estimate $a/b = 1{\cdot}78$. The solution was obtained by satisfying the constant-pressure condition in the neighbourhood of the crest and the point at infinity. The second work

N	A	F^2	a/b	$\|\tau\|$	a_2	a_4	a_6	a_8	a_{10}	a_{12}	a_{14}	a_{16}	a_{18}
1	−0·3242	1·6366	1·8156	1·61 %	−0·1667	—	—	—	—	—	—	—	—
3	−0·3106	1·6397	1·8212	0·59 %	−0·1463	0·0040	−0·0094	—	—	—	—	—	—
5	−0·3096	1·6470	1·8250	0·31 %	−0·1439	0·0018	−0·0072	0·0032	−0·0035	—	—	—	—
7	−0·3099	1·6512	1·8269	0·16 %	−0·1437	0·0009	−0·0062	0·0025	−0·0028	0·0018	−0·0019	—	—
9	−0·3106	1·6542	1·8281	0·12 %	−0·1441	0·0005	−0·0057	0·0022	−0·0024	0·0015	−0·0016	0·0012	−0·0012

TABLE 1

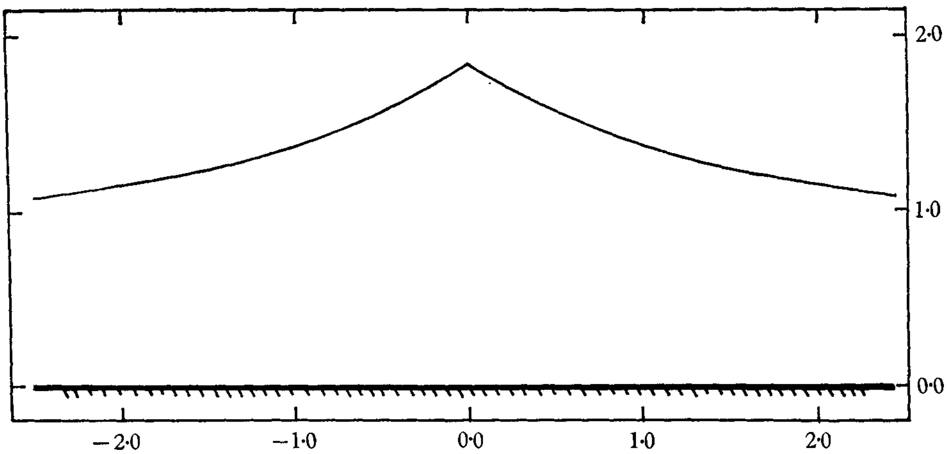

FIGURE 2. The wave profile: $N = 9$.

of interest is due to Packham (1952). Packham approximated the constant-pressure condition on the free streamline by replacing $\sin\theta$ by $l \sin 3\theta$, where $\theta = \arg(\zeta)$. For the new boundary conditions he obtained an exact solution with the condition

$$F^2 = (3l \tan \tfrac{1}{3}\pi)/(\tfrac{1}{3}\pi).$$

Packham determined l by minimizing the integral

$$\int_0^{\frac{1}{6}\pi} (\sin\theta - l\sin 3\theta)^2\, d\theta$$

to obtain $l = 3\sqrt{(3)}/(9\pi)$. By using this value one obtains

$$F^2 = 2{\cdot}05.$$

If one uses $l = \frac{1}{3}$, however, the Froude number $F^2 = 1{\cdot}6540$ is obtained, which is in remarkable agreement with those found in table 1. The writer believes that this agreement is not accidental. For $l = \frac{1}{3}$, the constant-pressure condition will be satisfied over the majority of the wave profile. Moreover, the singularity at the crest possessed by Packham's solution is independent of l. Hence the crest is of proper shape regardless of the l chosen. It appears then, that the upstream Froud number $F^2 = 1{\cdot}654$ is correct to four significant figures.

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